Bregje Pauwels **Cohom. of elementary abelian p-groups**

Talk 3, Day 3, ~~Talk~~

G group, $\mathrm{char}(k) = p$

- Recall: when G is a p-group, every proj. kG-mod is free.
- Let's start with $G = \mathbb{Z}/p = \langle g \mid g^p = 1 \rangle$ $\quad g-1 \leftrightarrow y$

  We can write $kG \cong k[y]/(y^p)$
- Can write a (fin. dim.) kG-module as a diagram:

  vertex $\longleftrightarrow$ element in k-basis for M

  arrow $\longleftrightarrow$ mult. by an element in $\mathrm{rad}\, kG \setminus \mathrm{rad}^2 kG$.

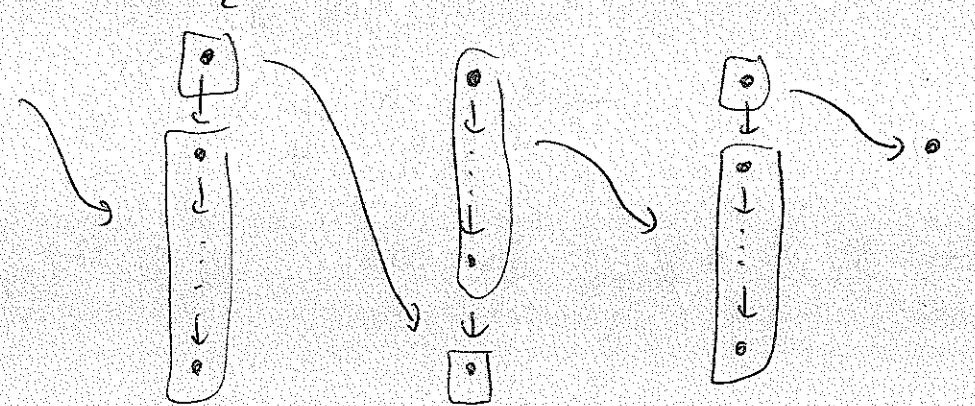
E.g.: $M = kG$ is:
$$\begin{array}{rl} 1 & \bullet \\ & \downarrow y \\ y & \bullet \\ & \downarrow y \\ y^2 & \bullet \\ & \vdots \\ & \bullet \\ & \downarrow y \\ y^{p-1} & \bullet \end{array}$$

k trivial kG-module: $\bullet\, 1$

k has a ~~trivial~~ (minimal) proj. resolution as follows:

$$\cdots \xrightarrow{y} \underset{\substack{\| \\ X_2}}{\frac{k[y]}{y^p}} \xrightarrow{y^{p-1}} \underset{\substack{\| \\ X_1}}{\frac{k[y]}{y^p}} \xrightarrow{y} \underset{\substack{\| \\ X_0}}{\frac{k[y]}{y^p}} \xrightarrow{\varepsilon} \underset{k}{k}$$

etc.



$G = \mathbb{Z}/p = \langle g \rangle$

<u>Theorem</u>: $$\boxed{H^\bullet(G,k) = \begin{cases} k[\gamma_1] & : p = 2 \\ k[\gamma_1,\gamma_2]/(\gamma_1^2) & : p > 2 \end{cases} \text{ with } |\gamma_i| = i.}$$

<u>Proof</u>:

- As a k-VS : $H^\bullet(G,k) = \bigoplus_{i=0}^{\infty} k\gamma_i$, $\gamma_i \in H^i(G,k) \setminus \{0\}$.

Apply $\mathrm{Hom}(-,k)$ to proj. res of k: (Rebecca did it):

$$0 \to \mathrm{Hom}(X_0,k) \xrightarrow{0} \mathrm{Hom}(X_1,k) \xrightarrow{0} \cdots$$

$$\cdots \xrightarrow{0} \mathrm{Hom}(X_i,k) \xrightarrow{0} \cdots$$

$\updownarrow$

generated by $\gamma_i : X_i = kG \longrightarrow k$

$1 \longmapsto 1$.

$\rightsquigarrow H^\bullet(G,k) = \bigoplus_{i=0}^{\infty} k\gamma_i$ ✓

- Ring structure?

Represent cohom. elements γ_i by chain maps (of deg $-i$) and then compose.

Chain map for $\gamma_2 : X_2 \longrightarrow k$:

$$\begin{tikzcd}
\cdots \arrow[r, "y^{p-1}"] & X_4 \arrow[r, "y"] & X_4 \arrow[r, "y^{p-1}"] \arrow[d, dashed, "id"] & X_3 \arrow[r, "y"] \arrow[d, dashed, "id"] & X_2 \arrow[r, "y"] \arrow[d, dashed, "id"] \arrow[dr, "\gamma_2"] & X_1 \arrow[r, "y^{p-1}"] & X_0 \arrow[r] & k \\
& \arrow[r] & X_2 \arrow[r, "y^{p-1}"'] & X_1 \arrow[r, "y"'] & X_0 \arrow[r, "\varepsilon"'] & k & &
\end{tikzcd}$$

$\Rightarrow \gamma_i \gamma_2 = \gamma_{i+2}$ for $i \geq 0$ $\in H^{i+2}(G,k)$.

• Chain map for $\gamma_1 : X_1 \to k$

$$\begin{tikzcd}
\cdots X_4 \arrow[r, "y^{p-1}"] \arrow[d, "y^{p-2}"] & X_3 \arrow[r, "y"] \arrow[d, "id"] & X_2 \arrow[r, "y^{p-1}"] \arrow[d, "y^{p-2}"] & X_1 \arrow[r, "y"] \arrow[d, "id"] \arrow[dr, "\gamma_1"] & X_0 \arrow[r, "\varepsilon"] & k \\
\arrow[r, "y"'] & \arrow[r, "y^{p-1}"'] & \arrow[r, "y"'] & X_0 \arrow[r] & k &
\end{tikzcd}$$

If $p = 2$ $\quad \gamma_i \gamma_1 = \gamma_{i+1} \quad (i \geq 0)$

If $p > 2$ $\quad \gamma_1^2 = 0 \quad$ in $H^2(G, k)$

$$X_2 \xrightarrow{y^{p-2}} X_1 \xrightarrow{\gamma_1} k$$
$$1 \longmapsto y^{p-2} \longmapsto 0$$

• Now $G = (\mathbb{Z}/p)^n = \langle g_1, \dots, g_n \rangle$

$H_i := \langle g_i \rangle$, so $G = H_1 \times \dots \times H_n$

$$\Rightarrow \quad kG = kH_1 \otimes \dots \otimes kH_n .$$

<u>Künneth theorem</u>: C_*, D_* complexes of kG-mod left-bounded, then $H_n(C_* \otimes D_*) \cong \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*)$ as kG-modules.

$\Rightarrow \quad P_* \otimes Q_* \xrightarrow{\alpha_1 \otimes \alpha_2} M \otimes N \quad$ is a proj res.

if $\left\{ \begin{array}{l} P_* \xrightarrow{\alpha_1} M \\ Q_* \xrightarrow{\alpha_2} N \end{array} \right.$ are proj. resolutions.

<u>Theorem</u>: Let $G = (\mathbb{Z}/p)^n$. Then

$$H^*(G, k) = \begin{cases} k[\xi_1, \dots, \xi_n] & p = 0 \\ k[\theta_1, \dots, \theta_n] \otimes \Lambda(\xi_1, \dots, \xi_n) & (p > 0) \end{cases}$$

with $|\xi_i| = 1$, $|\theta_i| = 2$

Here: $\Lambda(\xi_1, \dots, \xi_n) = k\langle \xi_1, \dots, \xi_n \rangle / (\xi_i^2, \; \xi_i \xi_j + \xi_j \xi_i)$

<u>Proof</u>: (min.) proj. res. for k as kH_i-module:

$$X_*^{(i)} \xrightarrow{\varepsilon_i} k$$

Künneth $\Longrightarrow$ $X_* = X_*^{(1)} \otimes \dots \otimes X_*^{(n)} \xrightarrow[\sim]{\varepsilon_1 \otimes \dots \otimes \varepsilon_n} k$, as a kG-module.

$$X_m = \bigoplus_{j_1 + \dots + j_n = m} \underbrace{X_{j_1}^{(1)} \otimes \dots \otimes X_{j_m}^{(m)}}_{kG}$$

$\mathrm{Hom}_{kG}(X_m, k)$: k-vs with basis $\gamma_{j_1}^{(1)} \otimes \dots \otimes \gamma_{j_m}^{(n)}$ for $j_1 + \dots + j_n = m$.

(Here, remember $\gamma_{j_i}^{(i)} \in H^{j_i}(H_i, k)$, represented by cocycles $kH_i = X_{j_i} \longrightarrow k$, $1 \longmapsto 1$

Can represent $\gamma^{(i)}_{j_i}$ as a chain map

$X^{(i)}_* \longrightarrow X^{(i)}_*$ of degree $-j_i$.

$$kH_i = k \otimes \cdots \otimes kH_i \otimes \cdots \otimes k$$

$\curvearrowleft y_i$ $\quad\curvearrowleft y_j$: acts as zero ~~trivially~~ for $j \neq i$

$$kG = k[y_1, \dots, y_n] / (y_1^p, \dots, y_n^p)$$

We can repr. every $\gamma^{(1)}_{j_1} \otimes \cdots \otimes \gamma^{(n)}_{j_n}$ by a product of these chain maps

$X_* \longrightarrow X_*$ of degree m.

- let
$$\begin{tikzcd} \xi_i : X_i \arrow[r, "\mathrm{proj}"] \arrow[dr] & X_0^{(1)} \otimes \cdots \otimes X_0^{(i-1)} \otimes X_1^{(i)} \otimes X_0^{(i+1)} \otimes \cdots \otimes X_0^{(n)} \arrow[d, "\varepsilon_1 \otimes \cdots \otimes \gamma_1^{(i)} \otimes \cdots \otimes \varepsilon_n"] \\ & k \end{tikzcd}$$

$$\begin{tikzcd} \theta_i : X_2 \arrow[r, "\mathrm{proj}"] \arrow[dr] & X_0^{(1)} \otimes \cdots \otimes X_0^{(i-1)} \otimes X_2^{(i)} \otimes X_0^{(i+1)} \otimes \cdots \otimes X_0^{(n)} \arrow[d, "\varepsilon_1 \otimes \cdots \otimes \gamma_2^{(i)} \otimes \cdots \otimes \varepsilon_n"] \\ & k \end{tikzcd}$$

{ can see...

- If $p = 2$, ξ_i generate $H^*(G, k)$ as k-algebra
- If $p > 2$, ξ_i, θ_j ——— " ———
- $\xi_i \xi_j = -\xi_j \xi_i$ because $H^*(G, k)$ is graded comm.

Remark

1) For $p>2$, $(\xi_1,\dots,\xi_n) = \mathrm{rad}\ H^*(G,k)$

$$\Rightarrow \quad H^*(G,k) / \mathrm{rad}\, H^*(G,k) \cong k[\theta_1,\dots,\theta_n] =: k[\underline{\theta}]$$

<u>for all p !</u>

Have a polynomial ring.

2) $H^*(G,k)$ is f.g. (free) module over $k[\underline{\theta}]$ generated by ξ_i's.

3) Recall: for a (complex of) kG-modules M, $H^*(G,M)$ is a graded right module over $H^*(G,k)$ hence over $k[\underline{\theta}]$.

Prop: $H^*(G,M)$ is fin-gen. over $k[\underline{\theta}]$ for $M \in D^f(kG)$.

Proof: - $D^f(kG) = \mathrm{thick}_{D^f(kG)}(k)$

- true for $H^*(G,k)$: f.g over $k[\underline{\theta}]$

$\Rightarrow$ get it for the whole cat., via $\oplus$-ands & cones: if $N' \to N \to N'' \to \Sigma N'$ exact Δ in $D^f(kG)$, $H^*(G,N') + H^*(G,N'')$ f.g.

L.E.S in coHom ... $\Longrightarrow$ $H^*(G,N)$ f.g. too. $\square$

In particular: f.g. $\Rightarrow$ noetherian

Def: $M \in D^b(kG)$, $V_G(M) := \operatorname{supp}_{H^*(G,k)}(H^*(G,M))$

mod kG

$\ast$

G a p-group

$\downarrow$

$\operatorname{supp}_{H^*(G,k)} \operatorname{Ext}^*(M,M) \equiv$

$(\mathrm{id}_M)_{\mathfrak{p}}$

$\operatorname{Ext}^*(M,M)_{\mathfrak{p}} \neq 0 \iff \operatorname{Ext}^*(X,M)_{\mathfrak{p}} \neq 0 \ \forall X$

$\iff \operatorname{Ext}^*(\text{simples}, M)_{\mathfrak{p}} \neq 0$

$R \curvearrowright \operatorname{End}^*(M,M)_{\mathfrak{p}} \stackrel{?}{=} 0$

$\mathcal{T} \ni M$

$\downarrow \quad \downarrow$

$\mathcal{T}_{\mathfrak{p}} \ni M \stackrel{?}{=} 0$

$\operatorname{Ext}^*(M,M)_{\mathfrak{p}} = 0 \iff \operatorname{Ext}^*(M,M)_{\mathfrak{p}} = 0$

$\Updownarrow$ G a p-group

$\operatorname{Ext}^*(k,M) = 0$